In order to calculate forces exerted by moving fluids, we must be able to describe the dynamics mathematically.

- To do this, we must be able to describe the motion (i.e., displacement, velocity, and acceleration) called kinematics.

2 Methods of Description

**Lagrangian** → keep track of individual, identifiable elements of fixed mass as a function of time (system approach)

- In fluid mechanics, this “particle tracking” approach would be a bookkeeping nightmare due to huge number of molecules in region of interest and small distance between molecular collisions.

Example: Air @ STP → \( l \approx 66 \text{ nm} \)

**Eulerian** → we draw a control surface that bounds a control volume. Flux of mass, momentum, energy, etc. across the boundary of control volume. Here, we keep track of points in space as a function of time \((x, y, z, t)\)

→ **field** approach
What is the difference between the Lagrangian vs. Eulerian descriptions?

- Consider a Cartesian coordinate system with vector notation of velocity \( \vec{V} \)
  \[
  \vec{V} = u \hat{i} + v \hat{j} + w \hat{k}
  \]

- **Lagrangian**: consider an individual fluid particle
  \[
  \vec{V}(t) = u(t) \hat{i} + v(t) \hat{j} + w(t) \hat{k}
  \]
  \[
  \vec{a}(t) = \frac{d}{dt} \vec{V}(t) = \frac{du(t)}{dt} \hat{i} + \frac{dv(t)}{dt} \hat{j} + \frac{dw(t)}{dt} \hat{k}
  \]

- **Eulerian**: consider a velocity field in space
  \[
  \vec{V}(x, y, z, t) = u(x, y, z, t) \hat{i} + v(x, y, z, t) \hat{j} + w(x, y, z, t) \hat{k}
  \]
  - via the chain rule
  \[
  \vec{a}(x, y, z, t) = \frac{d}{dt} \vec{V}(x, y, z, t) = \frac{\partial \vec{V}}{\partial t} + \frac{\partial \vec{V}}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial \vec{V}}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial \vec{V}}{\partial z} \frac{\partial z}{\partial t}
  \]
  \[
  = \frac{\partial \vec{V}}{\partial t} + u \frac{\partial \vec{V}}{\partial x} + v \frac{\partial \vec{V}}{\partial y} + w \frac{\partial \vec{V}}{\partial z}
  \]
  - local acceleration \( \to \) due to time variations at a particular point in space
  - convective acceleration \( \to \) due to spatial variations in velocity field
Recall the gradient operator in Cartesian coordinates:
\[ \nabla = \frac{\partial (\ )}{\partial x} \hat{i} + \frac{\partial (\ )}{\partial y} \hat{j} + \frac{\partial (\ )}{\partial z} \hat{k} \]

\[ a(x,y,z,t) = \frac{d\vec{V}(x,y,z,t)}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \]

- \( \vec{V} \cdot \nabla = u \frac{\partial (\ )}{\partial x} + v \frac{\partial (\ )}{\partial y} + w \frac{\partial (\ )}{\partial z} \)

- vector equation w/ 3 components: (one for each component of \( \vec{V} \))

\[
\begin{align*}
a_x(x,y,z,t) &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\
a_y(x,y,z,t) &= \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\
a_z(x,y,z,t) &= \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \\
\end{align*}
\]

- Convective acceleration term \((\vec{V} \cdot \nabla)\vec{V}\) is nonlinear!

- \( x, y, z, t \) are independent variables
- \( u, v, w \) are dependent variables
- Product of any two dependent variables (and or their derivatives) \( \rightarrow \) nonlinear term
• **Example:** \( u \frac{\partial u}{\partial x} \) is product of \( u \) and its partial x-derivative \( \frac{\partial u}{\partial x} \) \( \Rightarrow \) nonlinear

\[ \therefore \text{ Advantage:} \]
Eulerian method is simpler because it allows us to look at a region in space instead of forcing us to follow trillions of fluid molecules.

\[ \therefore \text{ Disadvantage:} \]
Eulerian method is more complicated because the convective term is nonlinear (greater mathematical complexity).
Example \( \vec{V}(x, y, z, t) = 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \). Find the local, convective, and total accelerations and \( x, y, z \) components of total acceleration.

Solution

1. Find \( u, v, w. \) \( u = 3t, \ v = xz, \ w = ty^2 \)

2. local acceleration: \( \frac{\partial \vec{V}}{\partial t} = \frac{\partial}{\partial t} \left[ u \hat{i} + v \hat{j} + w \hat{k} \right] = \frac{\partial}{\partial t} \left[ 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right] = 3 \hat{i} + y^2 \hat{k} \)

\[ (\vec{V} \cdot \vec{V}) \vec{V} = u \frac{\partial (\vec{V})}{\partial x} + v \frac{\partial (\vec{V})}{\partial y} + w \frac{\partial (\vec{V})}{\partial z} \]

3. convective acceleration:

\[ \frac{\partial}{\partial t} \left[ 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right] = 3t \frac{\partial}{\partial x} \left( 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right) + xz \frac{\partial}{\partial y} \left( 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right) + ty^2 \frac{\partial}{\partial z} \left( 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right) \]

\[ = 3t \frac{\partial}{\partial x} \left( 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right) + xz \frac{\partial}{\partial y} \left( 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right) + ty^2 \frac{\partial}{\partial z} \left( 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right) \]

\[ = 3t \left( \frac{\partial}{\partial x} \left( 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right) \right) + xz \left( \frac{\partial}{\partial y} \left( 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right) \right) + ty^2 \left( \frac{\partial}{\partial z} \left( 3t \hat{i} + xz \hat{j} + ty^2 \hat{k} \right) \right) \]

\[ = 3t (\hat{j}) + xz (\hat{j}) + ty^2 (\hat{j}) = (3t + xz + ty^2) \hat{j} + (2xyz) \hat{k} \]

4. total acceleration = local acceleration + convective acceleration

\[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{V}) \vec{V} = 3 \hat{i} + y^2 \hat{k} + (3t + ty^2) \hat{j} + (2xyz) \hat{k} \]

\[ = (3) \hat{i} + (3t + ty^2) \hat{j} + (y^2 + 2xyz) \hat{k} \]

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If $\vec{V} = \vec{V}(x, y, z)$ but not $\vec{V}(x, y, z, t)$, is $\ddot{a} = 0$?

- **Answer:** No, convective term will not be zero even if local acceleration is zero.

If $\vec{V} \neq \vec{V}(t)$, then $\frac{\partial \vec{V}}{\partial t} = 0 \rightarrow$ flow is steady.

If $\vec{V} = \vec{V}(t)$, then $\frac{\partial \vec{V}}{\partial t} \neq 0 \rightarrow$ flow is unsteady.

If $\vec{V}$ is a function of:
- only 1 spatial coordinate $\rightarrow$ 1-D flow (simplest case)
- only 2 spatial coordinates $\rightarrow$ 2-D flow
- all 3 spatial coordinates $\rightarrow$ 3-D flow (most difficult case)

<table>
<thead>
<tr>
<th>Example: $\vec{V} = Axy^3 e^{j\omega t} \hat{j}$</th>
<th>Example: $\vec{V} = 5t - \sin(\omega t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eulerian description since $\vec{V} \neq \vec{V}(\bar{x}, t)$</td>
<td>Lagrangian description since $\vec{V} \neq \vec{V}(\bar{x})$</td>
</tr>
<tr>
<td>2-D since function of $x$ and $y$</td>
<td>unsteady since function of $t$</td>
</tr>
</tbody>
</table>